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On the Observability of Autonomous Nonlinear Systems

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1. INTRODUCTION

In connection with state determination, the observability problem of nonlinear dynamical systems has been studied [1–5]. The observability problem is to find conditions under which a knowledge of the input–output data uniquely determines the state of the system.

Griffith and Kumar [4] treated this problem and reported some necessary conditions and a sufficient condition of observability of nonlinear dynamical systems. This paper treats autonomous nonlinear dynamical systems. In particular, for the construction of optimal control, it is assumed that the optimal control is a function of state. In this case, time-invariant nonlinear dynamical systems having inputs turn into time-invariant nonlinear dynamical systems without inputs, i.e., autonomous nonlinear dynamical systems. This paper classifies these autonomous nonlinear systems into three types of systems, that is, C^k type, analytic type, and polynomial type, and gives a necessary and sufficient condition of observability for each type of system.

2. STATEMENT OF THE PROBLEM

Throughout this paper it is assumed that the motion of an autonomous nonlinear system is described by the ordinary differential equation

$$\dot{x} = f(x) \tag{2.1}$$

and the output of the system is given by

$$y = h(x), \tag{2.2}$$

where x is a state n-column vector and y is an output r-column vector.

Assume $f(\cdot)$ has continuous partial derivatives on the real Euclidean n-space R^n. There exists a unique solution of (2.1) passing through x_0 at initial time t_0 on some interval containing t_0 in its interior [6, Chap. 1]. In an autonomous

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differential equation, by shifting the time axis, initial time t_0 may be equal to 0, which does not lose generality. From the continuation of solutions [6, p. 13], there exists a maximal interval $(m_1(x_0), m_2(x_0))$ that contains any interval on which there exists a solution of (2.1) passing through x_0 at $t_0 = 0$. Denote such a unique solution on the maximum intervals $(m_1(x_0), m_2(x_0))$ by $x(t; x_0)$ and the corresponding output $h(x(t; x_0))$ by $y(t; x_0)$.

The system (2.1), (2.2) is observable if there do not exist two different initial states x_1 and x_2 such that $y(t; x_1) = y(t; x_2)$ for any $t \in \bigcap_{i=1}^{2} [0, m_2(x_i))$.

The problem which is considered in this paper is to find necessary and sufficient conditions for the system (2.1), (2.2) to be observable.

It is assumed in Section 3 that f in (2.1) has continuous partial derivatives and continuous mixed partial derivatives up to and including order $k - 1$ on R^n, i.e., $f \in C^{k-1}(R_n)$, and $h \in C^k(R^n)$, where integer $k \geqslant 1$. We presume in Section 4 that f and h are analytic functions on R^n, and in Section 5 that f and h are polynomials in R^n.

The discussion is limited to a real scalar output for the sake of simplicity. Extension to a vector output is straightforward.

3. C^k Type of Systems

Denote the output $h(x(t; x_0))$ at t by $h_0(x(t))$; i.e.,

$$y(t) = h(x(t; x_0)) = h_0(x(t)).$$

From the assumptions $f \in C^{k-1}(R^n)$ and $h \in C^k(R^n)$, differentiating the above equation with respect to t, and taking account of (2.1), we form the recurrence relations

$$\begin{aligned} y(t) &= h_0(x(t)), \\ \frac{d}{dt} y(t) &= \frac{\partial h_0}{\partial x}(x(t)) f(x(t)) = h_1(x(t)), \\ \frac{d^2}{dt^2} y(t) &= \frac{d}{dt} y^{(1)}(t) = \frac{\partial h_1}{\partial x}(x(t)) f(x(t)) = h_2(x(t)), \\ \frac{dy^{(k-1)}(t)}{dt} &= \frac{\partial h_{k-2}}{\partial x}(x(t)) f(x(t)) = h_{k-1}(x(t)), \end{aligned} \tag{3.1}$$

where $(\partial h/\partial x)(x)$ is the Jacobian matrix of h at x. These relations can be represented as the following with a nonlinear mapping $H_k(\cdot)$.

$$z_k(t) = H_k(x(t)), \tag{3.2}$$

where

$$z_k(t) = \begin{bmatrix} y(t) \\ \vdots \\ y^{(k-1)}(t) \end{bmatrix}, \qquad H_k(x) = \begin{bmatrix} h_0(x) \\ \vdots \\ h_{k-1}(x) \end{bmatrix}.$$

Let us call this nonlinear mapping H_k the kth *observability mapping* of system (2.1), (2.2).

Denote the range of the function $H_k(\cdot)$ by $H_k(R^n)$. Suppose there exists the following output differential equation which has a unique solution on $H_k(R^n)$.

$$y^{(k)}(t) = \Phi(y(t), y^{(1)}(t),..., y^{(k-1)}(t)), \qquad t \geqslant 0, \tag{3.3}$$

where $\Phi(\cdot)$ maps $H_k(R^n)$ into R. Assume the system is observable. Then, the mapping H_k is univalent in R^n. (A one-to-one mapping is said to be univalent.) To see this, we can use the argument by contradiction. Assume there exist two different initial states x_1 and x_2 such that $H_k(x_1) = H_k(x_2)$. Then

$$\begin{bmatrix} y(0) \\ \vdots \\ y^{(k-1)}(0) \end{bmatrix} = z(0) = H_k(x_1) = H_k(x_2) \in H_k(R^n). \tag{3.4}$$

Also, the two outputs satisfy the same output differential equation (3.3). Their initial conditions are the same, from (3.4). Hence, the two outputs must be identically equal. This contradicts the fact that the system is observable. Conversely, if H_k is univalent, then by $(y(0),..., y^{(k-1)}(0))$ the initial state $x(0)$ of the system can be uniquely determined. Hence the system is observable.

The above discussion is summarized in the following proposition.

PROPOSITION 1. *Let $f \in C^{k-1}(R_n)$ and $h \in C^k(R^n)$. If the* kth *observability mapping $H_k(\cdot)$ is univalent in R^n, i.e., one-to-one from R^n onto $H_k(R^n)$, then the system* (2.1), (2.2) *is obserbable. Conversely, if the system* (2.1), (2.2) *is observable and there exists an output differential equation* (3.3) *of order k which has a unique solution on $H_k(R^n)$, then $H_k(\cdot)$ is univalent in R^n.*

EXAMPLE 1. This is Example 1 considered by Griffith and Kumar [4]. Consider $\dot{x}_1 = x_1$, $\dot{x}_2 = 2x_2$, and $y = x_1^2 + x_2^2$. Thus, $y^{(1)} = 2x_1^2 + 4x_2^2$ and $y^{(2)} = 4x_1^2 + 16x_2^2$. From a simple calculation, $y^{(2)} = 6y^{(1)} - 8y$. This is the output differential equation, which has a unique solution in R^2. Since $x_1^2 = (4y - y^{(1)})/2$ and $x_2^2 = (y^{(1)} - 2y)/2$, by Proposition 1 this system is not observable.

If $H_k(\cdot)$ is univalent in R^n, then there exists an n vector function $X_k(\cdot)$ uniquely defined on $H_k(R^n)$ such that

$$x = X_k(z) = X_k(H_k(x)). \tag{3.5}$$

From (2.1), (3.1), and the assumptions $f \in C^{k-1}(R^n)$ and $h \in C^k(R^n)$,

$$y^{(k)}(t) = \frac{\partial h_{k-1}}{\partial x}(x(t)) f(x(t)) = h_k(x(t)), \qquad t \geqslant 0. \tag{3.6}$$

Substituting $x(t) = X_k(z(t)) = X_k(y(t),..., y^{(k-1)}(t))$ into the right-hand side of (3.6) yields

$$y^{(k)}(t) = h_k(X_k(y(t),..., y^{(k-1)}(t))), \qquad t \geqslant 0, \tag{3.7}$$

which is the kth-order output differential equation of the system (2.1), (2.2). But it does not necessarily mean that this output differential equation has a unique solution. This is shown by the next example.

EXAMPLE 2. Consider $\dot{x} = 1$ and $y = x^3$. Since $H_1(x) = x^3$, $H_1(x)$ is univalent in R^1. Thus, this system is observable by Proposition 1. From $y = x^3$, $x = y^{1/3}$. Therefore $\dot{y} = 3x^2\dot{x} = 3y^{2/3}$. This output differential equation has two solutions for initial condition $y(0) = 0$. Those two solutions are $y(t) = 0$ and $y(t) = t^3$. On the other hand, $y^{(2)} = 6x$ and $y^{(3)} = 6$. Therefore, there exists the output differential equation $y^{(3)}(t) = 6$ and it has a unique solution for any initial condition.

4. ANALYTIC TYPE OF SYSTEMS

So far it has been assumed that $f \in C^{k-1}(R^n)$ and $h \in C^k(R^n)$. In this section, f and h are analytic functions on R^n. A vector-valued function f defined on a domain D of R^n is said to be analytic on D if each component f_i of f is representable by a convergent power series in some neighborhood of each point $x_0 \in D$.

PROPOSITION 2. *Let f and h in* (2.1), (2.2) *be analytic functions on R^n. The system* (2.1), (2.2) *is observable if and only if the equations $H_k(x_1) = H_k(x_2)$ $(k = 1, 2,...)$ imply only a trivial solution $x_1 = x_2$.*

Proof. Let x_0 be an initial state at $t_0 = 0$. From the fact that f is analytic on R^n, the solution $x(t; x_0)$ of (2.1) is analytic in the maximum domain interval $(m_1(x_0), m_2(x_0))$ [6, Chap. 1, Theorem 8.1]. Since h is a scalar analytic function on R^n, the output $y(t; x_0)$ is also analytic in $(m_1(x_0), m_2(x_0))$, and can be represented by the Taylor series

$$y(t; x_0) = \sum_{k=0}^{\infty} (1/k!)\, y^{(k)}(0; x_0)\, t^k \tag{4.1}$$

in some interval $|t| < \epsilon(x_0)$, $\epsilon(x_0) > 0$. Therefore, $y^{(k)}(0; x_1) = y^{(k)}(0; x_2)$ $(k = 0, 1, 2,...)$ or from (3.1), $H_k(x_1) = H_k(x_2)$ $(k = 1, 2,...)$ is equivalent to $y(t; x_1) = y(t; x_2)$ in interval $|t| < \min(\epsilon(x_1), \epsilon(x_2))$. From the analyticity of $y(t; x_i)$, this is equivalent to $y(t; x_1) = y(t; x_2)$ in $[0, \min(m_2(x_1), m_2(x_2))$. Thus, from the definition of observability, Proposition 2 is obtained.

Roughly speaking, Proposition 2 states that for the analytic type of systems infinite-derivatives data $y^{(k)}(0)$ $(k = 0, 1,...)$ are necessary to determine uniquely the initial state of the system. However, for the polynomial type of systems a stronger result can be obtained, that is, some finite-derivatives data $y^{(k)}(0)$ $(k = 0, 1,..., d)$ are sufficient to determine uniquely the initial state of the system. This is shown in the next section. Therefore, the problem is whether the same stronger result can be deduced for the analytic type of systems. This is impossible, which is indicated by the following proposition.

PROPOSITION 3. *Consider the system*

$$\begin{aligned} \dot{x} &= -x, \qquad x \in R, \\ y &= h(x), \end{aligned} \tag{4.2}$$

where

$$h(x) = x \prod_{k=1}^{\infty} (1 - e^{x^2 - k^2})^k. \tag{4.3}$$

Then,

(1) *the function h defined by* (4.3) *is analytic on R,*

(2) *the system* (4.2) *is observable, and*

(3) *for any integer $k \geqslant 1$, the equation $H_k(x) = H_k(0)$ has countably infinite solutions $x = 0$ and $x = \pm(k + j)$ $(j = 0, 1, 2,...)$.*

Before proceeding to the proof of Proposition 3, the following lemma, which concerns infinite products, is important to the proof of (1).

LEMMA 1. *Let $\{u_k(z)\}_{k=1,2,...}$ be a family of continuous complex-valued functions defined on a bounded, closed domain D of the complex space C. If the series $\sum_{k=1}^{\infty} |u_k(z)|$ is uniformly convergent in D, then the infinite product $\prod_{k=1}^{\infty} (1 + u_k(z))$ is uniformly convergent in D. Moreover, the infinite product $\prod_{k=1}^{\infty} (1 + u_k(z))$ is zero at $z \in D$ if and only if some factor $1 + u_k(z)$ is zero at $z \in D$.*

For the proof, see the elementary literature of infinite products or Ref. [7, Chaps. IV, VIII].

Proof of (1). Let the domain R of h extend to the complex space C, that is,

$$h(z) = z \prod_{k=1}^{\infty} (1 - e^{z^2 - k^2})^k, \qquad z \in C. \tag{4.3'}$$

In order to verify that h is analytic on R, it is sufficient to show that h is analytic on C. The infinite product of analytic functions is analytic, and if a sequence of analytic functions converges uniformly on a domain D of C, then the limit function is analytic on D. These are two basic properties of analytic functions. The series $\sum_{k=1}^{\infty} k \mid e^{z^2-k^2} \mid$ is uniformly convergent in C from the simple calculation

$$\frac{k+1}{k} \cdot \frac{\mid e^{z^2-(k+1)^2} \mid}{\mid e^{z^2-k^2} \mid} = \frac{k+1}{k} \cdot \mid e^{z^2-(k+1)^2-z^2+k^2} \mid$$
$$= \frac{k+1}{k} e^{-(2k+1)}.$$

Also, $1 - e^{z^2-k^2}$ is analytic on C. Therefore, using Lemma 1 together with the above two basic properties of analytic functions, $h(z)$ is analytic on C. From Lemma 1, all zero points of $h(z)$ on the real line R are $0, \pm 1, \pm 2,\ldots$, that is, integers. A simple calculation shows

$$\lim_{z\to k} \frac{h(z)}{(z^2 - k^2)^k} \neq 0 \qquad \text{for } k = 1, 2,\ldots.$$

This means that the order of zero point k of h is $\mid k \mid$, where $k = \pm 1, \pm 2,\ldots$. These results are necessary to verify (3).

Proof of (2). Let x_1 and x_2 be two initial states of the system (4.2). Each output of the system is given by $y(t\colon x_i) = h(x_i e^{-t})$, $t \geqslant 0$. If $h(x_1e^{-t}) = h(x_2e^{-t})$ for $t \geqslant 0$, then differentiating this equation with respect to t yields

$$h^{(1)}(x_1e^{-t})\, x_1e^{-t} = h^{(1)}(x_2e^{-t})\, x_2e^{-t}, \qquad t \geqslant 0.$$

Hence,

$$h^{(1)}(x_1e^{-t})\, x_1 = h^{(1)}(x_2e^{-t})\, x_2\,, \qquad t \geqslant 0.$$

Letting $t \to +\infty$ in the above equation gives

$$h^{(1)}(0)\, x_1 = h^{(1)}(0)\, x_2\,. \tag{4.4}$$

From (4.3′), the order of zero point $z = 0$ of h is 1, which implies $h^{(1)}(0) \neq 0$. Using this fact, (4.4) gives $x_1 = x_2$. This means that the system (4.2) is observable.

Proof of (3). From (4.2) and (3.1),

$$h_0(x) = h(x),$$
$$h_1(x) = -h^{(1)}(x)\, x,$$
$$h_2(x) = h^{(2)}(x)\, x^2 + h^{(1)}(x)\, x.$$

In general,

$$h_i(x) = (-1)^i h^{(i)}(x)\, x^i + \sum_{j=1}^{i-1} \alpha_{ij} h^{(j)}(x)\, x^j, \qquad j = 0, 1,..., \tag{4.5}$$

where each α_{ij} is real constant. Therefore, $h_i(0) = 0$ ($i = 0, 1,..$), and this means $H_k(0) = 0_k$ for each k (0_k denotes k-dimensional zero vector). From the fact that the order of zero point of k is $|\, k \,|$ ($k = \pm 1, \pm 2,...$), it is seen that $h^{(k)}(\pm k) \neq 0$ and $h^{(k)}(\pm(k+j)) = 0$ ($k = 1, 2,..., j = 1, 2, 3,...$). This means from (4.5) that $h_k(\pm k) \neq 0$ and $h_k(\pm(k+j)) = 0$ ($k = 1, 2,...,$ and $j = 1, 2,.$)... This and the fact that all solutions of $h(x) = 0$, $x \in R$, are integers yield that for each integer $k \geqslant 1$ the equation $H_k(x) = 0_k$ has the solutions $x = 0$ and $x = \pm(k+j)$ ($j = 0, 1, 2,...$).

5. Polynomial Type of Systems

In this section, f and h in (2.1), (2.2) are assumed to be polynomials on R^n. A vector-valued function on R^n is said to be a polynomial on R^n if each component f_i of f is a polynomial in components $x_1, x_2,..., x_n$ of $x \in R^n$ with real scalar coefficients. Then, a stronger result can be obtained than those in preceding sections, that is,

Theorem 1. *Let f and h in* (2.1), (2.2) *be polynomials on R^n. The system* (2.1), (2.2) *is observable if and only if there exists some integer $d \geqslant 1$ such that the* dth *observability mpping H_d is univalent in R^n.*

Before proceeding to the proof of Theorem 1, some preliminary results are required. We denote the set of all scalar-valued polynomials on R^n by $R[x_1,..., x_n]$, or simply $R[x]$. Then, $R[x]$ is a commutative ring with a unit element. Let A be a commutative ring with a unit element. A subset I of A is called *ideal* of A if for any $a_1, a_2 \in A$ and $i_1, i_2 \in I$, it holds that $a_1 i_1 + a_2 i_2 \in I$. A ring A is called *Noetherian* if for any increasing ideal sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of A there exists a number m such that $I_m = I_{m+1} = I_{m+2} = \cdots$, or equivalently, any ideal I of A can be generated by a finite number of elements of I.

Lemma 2 (Hilbert's basis theorem). *For any integer $n > 0$, the polynomial ring $R[x_1,..., x_n]$ is Noetherian* [8, p. 143].

Proof of Theorem 1. From Proposition 2, it is sufficient to verify the "only-if" part. For $x, y \in R^n$, let

$$p_i(x, y) = h_i(x) - h_i(y), \qquad \text{for } i = 0, 1, 2,.... \tag{5.1}$$

Then each $p_i(x, y)$ belongs to $R[x, y]$. For the polynomial sequence

$\{p_i(x, y)\}_{i=0,1,\ldots}$, let us consider the polynomial subset sequence $\{I_k\}$ such that each I_k is generated by k polynomials $p_0, p_2, \ldots, p_{k-1}$, that is,

$$I_k = \left\{ p \in R[x, y] \mid p = \sum_{i=0}^{k-1} a_i p_i,\ a_i \in R[x, y] \right\}. \tag{5.2}$$

Then $\{I_k\}$ is an increasing ideal sequence of $R[x, y]$. Therefore, by Hilbert's basis theorem, there exists a number $d \geqslant 1$ such that $I_d = I_{d+1} = I_{d+2} = \cdots$. Hence for any integer $k \geqslant d$, polynomial $p_k(x, y)$ is represented by

$$p_k(x, y) = \sum_{i=0}^{d-1} a_{ki}(x, y)\, p_i(x, y), \qquad a_{ki}(\cdot) \in R[x, y]. \tag{5.3}$$

From (5.1) and (5.3), for any integer $k \geqslant d$

$$h_k(x) - h_k(y) = \sum_{i=0}^{d-1} a_{ki}(x, y)\, [h_i(x) - h_i(y)], \qquad a_{k,i}(\cdot) \in R[x, y]. \tag{5.4}$$

If the system is observable, by Proposition 2, countably infinite equations $h_k(x) = h_k(y)$ $(k = 0, 1, 2,\ldots)$ imply $x = y$. Since from (5.4) countably infinite equations $h_k(x) = h_k(y)$ $(k = 0, 1, 2,\ldots)$ are equivalent to d equations $h_k(x) = h_k(y)$ $(k = 0,\ldots, d-1)$, if the system is observable, then d equations $h_k(x) = h_k(y)$ $(k = 0, 1,\ldots, d-1)$ imply $x = y$, that is, $H_d(x) = H_d(y)$ implies only trivial solution $= y$. This completes the proof.

By the proof of Theorem 1, in the case where f and h in (2.1), (2.2) are polynomials, the problem of observability of systems is reduced to two problems. The first is to find integer d such that equation $H_d(x) = H_d(y)$ is equivalent to countably infinite equations $H_{d+i}(x) = H_{d+i}(y)$ $(i = 0, 1, 2,\ldots)$. The second is to examine whether $H_d(x)$ is univalent in R^n. It seems to be difficult to treat the first problem for the general polynomial type of systems. In the case, however, where f in (2.1) is a first-degree polynomial, the first problem is easily solved. This is shown by the following theorem.

THEOREM 2. *Consider*

$$\dot{x} = Ax + b, \qquad y = p_m(x), \tag{5.5}$$

where A is an $n \times n$ matrix, b is an n-vector and $p_m(x)$ is a polynomial of degree m. Then there exists an output linear differential equation of order d

$$y^{(d)}(t) = \sum_{i=0}^{d-1} \alpha_i y^{(i)}(t), \qquad t \geqslant 0, \tag{5.6}$$

where α_i are real constants and order d satisfies

$$d \leqslant d_{n,m} = \sum_{i=0}^{m} ((n+i-1)!/i!(n-1)!). \tag{5.7}$$

Before taking up the proof of Theorem 2, the following lemma is necessary.

LEMMA 3. *The set of polynomials of degree m defined on R^n can be considered as a linear space of dimension $d_{n,m}$ given by*

$$d_{n,m} = \sum_{i=0}^{m} ((n+i-1)!/i!(n-1)!).$$

Proof. A polynomial of degree m defined on R^n can be expressed by

$$p(x_1, x_2, ..., x_n) = \sum a_{i_1, i_2, ..., i_n} x_1^{i_1} x^{i_2} \cdots x^{i_n}, \tag{5.8}$$

where the summation is over the set of n distinct integers $i_1, i_2, ..., i_n$ which satisfy

$$i_1 + i_2 + \cdots + i_n \leqslant m, \qquad i_j \geqslant 0. \tag{5.9}$$

The polynomial (5.8) is uniquely determined by its coefficients. The number M of those coefficients $a_{i_1, ..., i_n}$ is equal to that of different solutions of inequalities (5.9). The number of different integer solutions of the equation

$$i_1 + i_2 + \cdots + i_n = i, \qquad i_j \geqslant 0, \tag{5.10}$$

is given by

$$A_{i,n} = (n+i-1)!/i!(n-1)!. \tag{5.11}$$

(See [9, p. 36].) Hence

$$M = \sum_{i=0}^{m} A_{i,n}. \tag{5.12}$$

The set $\{a_{i_1}, ..., a_{i_n}\}$ can be considered as coordinates for a polynomial in the space of mth-degree polynomials. Therefore,

$$d_{n,m} = M = \sum_{i=0}^{m} A_{i,n} = \sum_{i=0}^{m} ((n+i-1)!/i!(n-1)!).$$

This completes the proof.

Proof of Theorem 2. From (5.5) and (3.1), it is evident that polynomials $h_0(x)$, $h_1(x)$, $h_2(x)$,... belong to the set of polynomials of degree m defined on R^n. The number of linear independent polynomials among $h_0(x)$, $h_1(x)$, $h_2(x)$,... is less than or equal to dimension $d_{n,m}$. Hence there exists some number $d \leqslant d_{n,m}$ such that

$$h_d(x) = \sum_{i=0}^{d-1} \alpha_i h_i(x), \tag{5.13}$$

where α_i are real constants. From (3.1), (5.13) implies (5.6), which completes the proof.

Remark. If constant term b is a zero vector in (5.5), from (5.5) and (3.1) no polynomial $h_i(x)$ contains a constant term. Therefore, in this case the summation $\sum_{i=0}^{m}$ in (5.7) can be replaced by the summation $\sum_{i=1}^{m}$.

Now consider the second problem mentioned above. This is to investigate whether the solution of a system of polynomials

$$z_i = p_i(x_1, \ldots, x_n), \qquad i = 1, 2, \ldots, d,$$

is unique. If the number d of equations is equal to the number n of variables, there are several sufficient conditions to assure the uniqueness of solutions [10, Chap. 7, or 11, Chap. 4]. However, in general, number d is greater than number n, and these sufficient conditions are not applicable. The task of demonstrating the uniqueness of solutions in this case is more difficult and must be studied for each system. In the remainder of this section, consider the special system given by

$$\dot{x} = Ax, \qquad y = c^{\mathrm{T}}x + x^{\mathrm{T}}Dx, \tag{5.14}$$

where A is an $n \times n$ matrix, c is an n column vector, D is an $n \times n$ symmetric matrix, and superscript T means the transpose of a vector or a matrix. Then from (3.1) and (5.14)

$$\begin{aligned} h_0(x) &= c^{\mathrm{T}}x + x^{\mathrm{T}}Dx, \\ h_1(x) &= c^{\mathrm{T}}Ax + 2x^{\mathrm{T}}DAx, \\ h_2(x) &= c^{\mathrm{T}}A^2x + 2x^{\mathrm{T}}A^{\mathrm{T}}DAx + 2x^{\mathrm{T}}DA^2x. \end{aligned}$$

In general,

$$h_i(x) = c^{\mathrm{T}}A^ix + \sum_{j+k=i, 0\leqslant j\leqslant k} \alpha_{jk}x^{\mathrm{T}}(A^j)^{\mathrm{T}}\, DA^kx \qquad \text{for } i = 0, 1, 2, \ldots, \tag{5.15}$$

where α_{jk} are real constants. From (5.15), an elementary calculation derives

$$H_d(x_1) - H_d(x_2) = \frac{\partial H_d}{\partial x}\left(\frac{x_1 + x_2}{2}\right)(x_1 - x_2), \qquad x_1, x_2 \in R^n. \tag{5.16}$$

From (5.16) the next theorem is obtained.

THEOREM 3. *The system* (5.14) *is observable if and only if the rank of the Jacobian matrix of* $H_d(x)$ *is equal to* n *for all* $x \in R^n$, *that is,*

$$\operatorname{rank} \frac{\partial H_d}{\partial x}(x) = n \qquad \textit{for all } x \in R^n, \tag{5.17}$$

where $d = d_{n,2} - 1 = n(n+3)/2$.

Proof. From Theorem 2 and its remark, the system (5.14) is observable if and only if the observability mapping $H_d(x)$ is univalent in R^n. If (5.17) holds, $H_d(x)$ is univalent in R^n from (5.16). Conversely, if (5.17) is violated, there exist a vector a and a vector h not zero such that

$$(\partial H_d/\partial x)(a)\, h = 0. \tag{5.18}$$

Let

$$x_1 = (2a+h)/2, \qquad x_2 = (2a-h)/2; \tag{5.19}$$

then,

$$(x_1 + x_2)/2 = a, \qquad x_1 - x_2 = h \neq 0. \tag{5.20}$$

Therefore, from (5.16), (5.18), and (5.20), it holds that $H_d(x_1) = H_d(x_2)$ and $x_1 \neq x_2$, which means that the mapping $H_d(x)$ is not univalent in R^n. The theorem is thus proved.

The following corollary is a direct result of Theorem 3.

COROLLARY. *If the system* (5.14) *is observable, then*

$$\operatorname{rank} \begin{bmatrix} c^{\mathrm{T}} \\ c^{\mathrm{T}}T \\ \vdots \\ c^{\mathrm{T}}A^{n-1} \end{bmatrix} = n; \tag{5.21}$$

that is, the linear system $\dot{x} = Ax$, $y = c^{\mathrm{T}}x$ *is observable.*

Proof. From (5.15),

$$\frac{\partial H_d}{\partial x}(0) = \begin{bmatrix} c^{\mathrm{T}} \\ c^{\mathrm{T}}A \\ \vdots \\ c^{\mathrm{T}}A^{d-1} \end{bmatrix}. \tag{5.22}$$

Since $d \geqslant n$, using the Cayley–Hamilton theorem,

$$\operatorname{rank} \begin{bmatrix} c^{\mathrm{T}} \\ c^{\mathrm{T}}T \\ \vdots \\ c^{\mathrm{T}}A^{d-1} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} c^{\mathrm{T}} \\ c^{\mathrm{T}}A \\ \vdots \\ c^{\mathrm{T}}A^{n-1} \end{bmatrix}. \tag{5.23}$$

Hence, from Theorem 3, we obtain the corollary.

Remark. Apply the above corollary to the system given in Example 1. Since the expression of output y does not contain the first-degree terms, this system is not observable.

6. Conclusions

This paper has considered the problem of observability of autonomous nonlinear dynamical systems. It has classified these systems into three type of systems, and has shown a necessary and sufficient condition of observability for each type of system. In engineering practice, for sufficiently smoothly varying systems, it is sufficient to use the third type of systems, i.e., the polynomial type of systems. However, the two problems mentioned in Section 5 are left unsolved for the more general polynomial type of systems.

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